The bar $\overline{}$ denotes the irreducible part of a tensor. The abbreviation

$$S_n = \left[S(S+1) - \frac{n}{2}\left(\frac{n}{2}+1\right)\right]^{1/2} \quad (A.14)$$

has been used. Thus one obtains

$$P_l^{SS} = (S\,0, l\,0 \,|\, S\,0)\,\frac{(2\,l-1)!!}{l!}\,\frac{1}{S_0 S_1 \dots S_{l-1}} \cdot \overline{s_{\mu_1} \dots s_{\mu_l}}\,\hat{x}_{\mu_1} \dots \hat{x}_{\mu_l}\,. \quad (A.15)$$

For $l=2$, using $(S\,0, 2\,0 \,|\, S\,0) = -S_0/2\,S_1$, one recovers (3.5),

$$P_2^{SS} = -\frac{3}{4\,S_1{}^2}\,\overline{s_\mu s_\nu}\,\hat{x}_\mu \hat{x}_\nu\,. \quad (A.16)$$

Likewise, with $(S\,0, 4\,0 \,|\, S\,0) = 3\,S_0 S_2/8\,S_1 S_3$, one finds for $l=4$

$$P_4^{SS} = \frac{105}{64}\,\frac{1}{S_1{}^2 S_3{}^2}\,\overline{s_{\mu_1} \dots s_{\mu_4}}\,\hat{x}_{\mu_1} \dots \hat{x}_{\mu_4}\,. \quad (A.17)$$

Influence of a Magnetic Field on the Brownian Motion of Particles with Magnetic Moment

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The influence of a magnetic field on the diffusion of Brownian particles with a magnetic moment parallel to their internal angular momentum is discussed. Starting point is a generalized Fokker-Planck equation. Application of the moment method leads to a set of transport-relaxation equations. From them the diffusion tensor depending on the external field is inferred.

In a previous paper [1] the Brownian motion of (spherical) rotating particles has been studied on the basis of a generalized Fokker-Planck equation. Due to the coupling of the translational and rotational motions, a diffusion flow gives rise to a correlation between linear and angular velocities. This correlation, in turn, influences the value of the diffusion coefficient.

In this paper, it is assumed that the (neutral) Brownian particles have a magnetic moment parallel to their internal angular momentum. Then, in the presence of an external magnetic field $\boldsymbol{H} = H\,\boldsymbol{h}$ (where $\boldsymbol{h}$ is a unit vector) the Brownian particles undergo a precessional motion with frequency ω_{H} which is equal to the gyromagnetic ratio times the magnitude H of the field. By this precessional motion the correlation between linear and angular velocities, existing in the transport situation without field, is partially destroyed. Consequently, the diffusion coefficient becomes a field-dependent second rank tensor. It is characterized by three scalar coefficients depending on the magnitude of the field. The magnetic field dependence of the diffusion is similar to the influence of the magnetic field on the transport properties of dilute polyatomic gases (SENFTLEBEN-BEENAKKER effect [2]).

Firstly, we shall state the generalized Fokker-Planck equation in which the precessional motion of the internal angular momentum is taken into account. Then the transport-relaxation equations needed for the discussion of the diffusion problem in the presence of a magnetic field are given. Finally the diffusion tensor is inferred from these equations.

Generalized Fokker-Planck Equation

An ensemble of Brownian particles is described by the distribution function

$$F = F(t, \boldsymbol{x}, \boldsymbol{V}, \boldsymbol{W})\,. \quad (1)$$

Here t is the time, $\boldsymbol{x}$ the position vector, and $\boldsymbol{V}$ is the velocity of a particle (in units of a thermal velocity v_0). The difference between the actual internal angular velocity and the angular velocity due to a thermal equilibrium polarization in a magnetic field (both in units of a thermal angular velocity) is de-

[1] S. HESS, Z. Naturforsch. **23 a**, 597 [1968].
[2] H. SENFTLEBEN, Phys. Z. **31**, 822, 961 [1930]. — J. J. M. BEENAKKER et al., Phys. Letters **2**, 5 [1962]. For a list of the literature on the SENFTLEBEN-BEENAKKER effect see BEENAKKER's review article in: Festkörperprobleme VIII, ed. O. MADELUNG, Vieweg, Braunschweig 1968.

noted by $\boldsymbol{W}$. This choice of the variable $\boldsymbol{W}$ has the advantage that, in the presence of a magnetic field, the Fokker-Planck operator has the same form as in the field free case. By

$$F(t,x,\boldsymbol{V},\boldsymbol{W}) = F_0(\boldsymbol{V},\boldsymbol{W})(1+\Phi(t,x,\boldsymbol{V},\boldsymbol{W})) \quad (2)$$

we define the quantity Φ which characterizes the deviation of the distribution function F from its equilibrium value F_0 (with temperature T_0 and number density n_0).

In the presence of a magnetic field the Fokker-Planck equation for Φ reads [3]

$$\frac{\partial \Phi}{\partial t} + \sqrt{\frac{2}{3}}\, v_0\, \boldsymbol{V} \cdot \frac{\partial \Phi}{\partial \boldsymbol{x}} - \omega_{\mathrm{H}}\, \boldsymbol{h} \cdot \left(\boldsymbol{W} \times \frac{\partial \Phi}{\partial \boldsymbol{W}}\right) + \Omega(\Phi) = 0. \quad (3)$$

The third term of Eq. (3) describes the precessional motion caused by the magnetic field. For zero magnetic field ($\omega_{\mathrm{H}} = 0$) this equation reduces to Eq. (1.17) of Ref. [1]. The generalized Fokker-Planck "collision" operator Ω contains three constants (reciprocal times) characterizing the interaction between the Brownian particle and the bulk fluid. Two of them (ω_1 and ω_2) are the friction coefficients for the translational and rotational motions while the third (ω_0) determines the "strength" of the coupling between translational and rotational motions. This coupling stems from a transverse force [4] which is akin to the Magnus effect. All three coefficients ω_0, ω_1, ω_2 are assumed to be independent of the magnetic field.

The ratio ω_0/ω_1 can be expected to be rather small [1]. Hence terms of order higher than $(\omega_0/\omega_1)^2$ may be neglected. Then it is sufficient to consider the transport relaxation equations for the first few moments (i. e. moments of expansion tensors up to "2^nd^ power") which shall be given below.

Transport Relaxation Equations

By applying WALDMANN's moment method [5] to the Fokker-Planck equation one obtains the transport-relaxation equations for certain mean values of interest. Inclusion of the first two vector moments $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ proved sufficient for the evaluation of the diffusion constant without a magnetic field [1]. Here $\boldsymbol{a}^{(1)}$ is proportional to the particle flux $\boldsymbol{j}$, $\boldsymbol{a}^{(2)}$ called "azimuthal polarization" is proportional to the angular momentum (spin) flux tensor. In the presence of a magnetic field the equations for the three irreducible parts of the spin flux tensor are coupled. Hence we now have to consider also the pseudo-scalar $b^{(1)}$ and second rank pseudo tensor $\mathbf{b}^{(1)}$ which are proportional, respectively, to the Cartesian trace and the second rank irreducible part of the spin flux tensor.

A gradient of the number density n (i. e. grad $a^{(1)}$) of the Brownian particles shall be the only "thermodynamical force" present. Then the necessary transport relaxation equations, as far as scalars and vectors are concerned, are

$$\frac{\partial b^{(1)}}{\partial t} - \sqrt{\frac{2}{3}}\, \omega_{\mathrm{H}}\, \boldsymbol{h} \cdot \boldsymbol{a}^{(2)} + (\omega_1 + \omega_2)\, b^{(1)} = 0, \quad (4)$$

$$\frac{\partial \boldsymbol{a}^{(1)}}{\partial t} + \frac{v_0}{3} \sqrt{3}\, \nabla a^{(1)} + \omega_1\, \boldsymbol{a}^{(1)} + \omega_0\, \boldsymbol{a}^{(2)} = 0, \quad (5)$$

$$\frac{\partial \boldsymbol{a}^{(2)}}{\partial t} + \sqrt{\frac{2}{3}}\, \omega_{\mathrm{H}}\, \boldsymbol{h}\, b^{(1)} + \tfrac{1}{2}\, \omega_{\mathrm{H}}\, \boldsymbol{h} \times \boldsymbol{a}^{(2)} - \omega_{\mathrm{H}} \frac{1}{\sqrt{2}} \mathbf{b}^{(1)} \cdot \boldsymbol{h} - \omega_0\, \boldsymbol{a}^{(1)} + (\omega_1 + \omega_2)\, \boldsymbol{a}^{(2)} = 0. \quad (6)$$

Here ∇ is the Nabla (gradient) operator. Eq. (6) shows that the azimuthal spin $\boldsymbol{a}^{(2)}$ undergoes a precessional motion with frequency $\frac{1}{2}\,\omega_{\mathrm{H}}$. In order to have a closed set of equations, one needs an additional equation, namely, that for the pseudo-tensor $\mathbf{b}^{(1)}$. However, since the latter occurs in (6) only in the vector combintation $\tilde{\boldsymbol{a}} = \mathbf{b}^{(1)} \cdot \boldsymbol{h}$, de facto one needs only the (simpler) equation for this vector $\tilde{\boldsymbol{a}}$:

$$\frac{\partial \tilde{\boldsymbol{a}}}{\partial t} + \frac{1}{\sqrt{2}}\, \omega_{\mathrm{H}} (\tfrac{1}{2}\, \boldsymbol{a}^{(2)} + \tfrac{1}{6}\, \boldsymbol{h}\, \boldsymbol{h} \cdot \boldsymbol{a}^{(2)}) + \tfrac{1}{2} \omega_{\mathrm{H}}\, \boldsymbol{h} \times \tilde{\boldsymbol{a}} + (\omega_1 + \omega_2)\, \tilde{\boldsymbol{a}} = 0. \quad (7)$$

The same set of transport-relaxation equations occurs in connection with the diffusion of a Lo-

[3] In order to treat the spin relaxation and the Senftleben-Beenakker effect of a dilute (model) gas consisting of rough spheres with a magnetic moment parallel to their internal angular velocity, L. WALDMANN suggested amending the Boltzmann equation firstly by adding a term which accounts for the precessional motion of the magnetic moments and secondly, by replacing the angular velocity by the difference between the angular velocity and the mean thermal equilibrium angular velocity due to the external magnetic field; cf., J. HALBRITTER, Diplomarbeit, Erlangen 1966. Apart from the explicit meaning of the collision operator Ω, Eq. (3) has the same form as the aforementioned Boltzmann equation.

[4] S. I. RUBINOW and J. B. KELLER, J. Fluid Mech. **11**, 447 [1961]. — S. HESS, Z. Naturforsch. **23 a**, 1095 [1968].

[5] L. WALDMANN, Z. Naturforsch. **18 a**, 1033 [1963].

rentzian gas of spin $\frac{1}{2}$ particles [6, 7]. These equations are of course also very similar to the transport relaxation equations for a dilute gas of particles with spin [8].

Diffusion Tensor

The constitutive law for the particle flux $\boldsymbol{j}$ in the presence of a magnetic field and the diffusion tensor may be inferred from the Eqs. (4) – (7) with all time derivatives put equal to zero (steady state). Solving these equations for $\boldsymbol{a}^{(1)} \sim \boldsymbol{j}$ expressed by grad $a^{(1)} \sim$ grad n one obtains [7]

$$\boldsymbol{j} = -D_{\|}\, \boldsymbol{h}\,\boldsymbol{h}\cdot\nabla n - D_{\perp}(\nabla - \boldsymbol{h}\,\boldsymbol{h}\cdot\nabla)\, n - D_{\text{trans}}\, \boldsymbol{h}\times\nabla n\,. \tag{8}$$

The diffusion constants for the cases where the magnetic field is parallel to or perpendicular to the concentration gradient ∇n are denoted by $D_{\|}$ and $D_{\perp}$, respectively. The term containing D_{trans} leads to a transverse particle flux perpendicular to both the concentration gradient and the external magnetic field. These three scalar diffusion coefficients are related to the zero field diffusion constant (k: Boltzmann's constant, m: mass of a Brownian particle)

$$D = \frac{k\,T_0}{m\,\omega_1}(1-A) \tag{9}$$

by

$$D_{\|} = D\left(1 + A\,\frac{\varphi^2}{1+\varphi^2}\right), \tag{10}$$

$$D_{\perp} = D\left(1 + \frac{1}{2}\,A\,\frac{\varphi^2}{1+\varphi^2}\right), \tag{11}$$

$$D_{\text{trans}} = D\,\tfrac{1}{2}\,A\,\frac{\varphi}{1+\varphi^2}\,. \tag{12}$$

Here the abbreviation

$$A = \frac{\omega_0^{\,2}}{\omega_1(\omega_1+\omega_2)} \tag{13}$$

has been used. The angle

$$\varphi = \omega_{\text{H}}/(\omega_1+\omega_2)$$

denotes the number of precessions which the magnetic moment of a particle undergoes during an effective relaxation time $\tau = (\omega_1+\omega_2)^{-1}$. Terms of higher than 2nd order in ω_0/ω_1 have been neglected in Eqs. (9) – (12).

Clearly, $D_{\|}$ and $D_{\perp}$ are equal to D at zero magnetic field. A magnetic field increases the diffusion constant. This is contrary to the SENFTLEBEN-BEENAKKER effect [2] where the transport constants are lowered by the magnetic field. For a strong magnetic field ($\varphi \to \infty$) $D_{\|}$ and $D_{\perp}$ reach their saturation values $D(1+A)$ and $D(1+\frac{1}{2}A)$, respectively. The transverse diffusion constant D_{trans} is zero for both $\varphi = 0$ and $\varphi \to \infty$, for $\varphi = 1$ it reaches the maximum value $\frac{1}{4}\,D\,A$.

Concluding Remarks

The relative change of the diffusion coefficient caused by an external magnetic field is at most equal to A [defined by (13)]. This quantity A can be expected to be rather small [1] (presumably less than 10^{-2} for particles with a radius of the order 10^{-6} cm). Since the diffusion constant cannot be measured as accurately as the heat conductivity or the viscosity (where differential methods are available) it will be difficult to detect experimentally an influence of an external magnetic field on the diffusion of Brownian particles. However, the simple model for Brownian motion of rotating particles presented here might give some hints for a kinetic theory of polyatomic liquids [9].

It is interesting to note that an influence of a magnetic field on the transport properties of solutions and liquids has been conjectured and looked for [10] long before SENFTLEBEN's measurements with gases. In contrast to liquid crystals [10, 11] – to the author's knowledge – up to now no effect has been found with liquids [10, 12]. However, it would be interesting to have further measurements with higher sensitivity.

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[6] L. WALDMANN and H. D. KUPATT, Z. Naturforsch. **18 a**, 86 [1963]. In this paper the second rank tensor $\mathbf{b}^{(1)}$ had been omitted. For an amended treatment see Ref. [7].

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